

A CRITICAL REGULARITY CONDITION ON THE ANGULAR VELOCITY OF AXIALLY SYMMETRIC NAVIER-STOKES EQUATIONS

QI S. ZHANG

ABSTRACT. Let v be the velocity of Leray-Hopf solutions to the axially symmetric three-dimensional Navier-Stokes equations. It is shown that v is regular if the angular velocity v_θ satisfies an integral condition which is critical under the standard scaling. This condition allows functions satisfying

$$|v_\theta(x, t)| \leq \frac{C}{r|\ln r|^{2+\epsilon}}, \quad r < 1/2,$$

where r is the distance from x to the axis, C and ϵ are any positive constants.

Comparing with the critical a priori bound

$$|v_\theta(x, t)| \leq \frac{C}{r}, \quad 0 < r \leq 1/2,$$

our condition is off by the log factor $|\ln r|^{2+\epsilon}$ at worst. This is inspired by the recent interesting paper [2] where H. Chen, D. Y. Fang and T. Zhang establish, among other things, an almost critical regularity condition on the angular velocity. Previous regularity conditions are off by a factor r^{-1} .

The proof is based on the new observation that, when viewed differently, all the vortex stretching terms in the 3 dimensional axially symmetric Navier-Stokes equations are critical instead of supercritical as commonly believed.

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1. INTRODUCTION

In rectangular coordinates, the incompressible Navier-Stokes equations are

$$(1.1) \quad \Delta v - (v \cdot \nabla)v - \nabla p - \partial_t v = 0, \quad \operatorname{div} v = 0,$$

where $v = (v_1(x, t), v_2(x, t), v_3(x, t)) : \mathbb{R}^3 \times [0, T] \rightarrow \mathbb{R}^3$ is the velocity field and $p = p(x, t) : \mathbb{R}^3 \times [0, T] \rightarrow \mathbb{R}$ is the pressure. In cylindrical coordinates r, θ, x_3 with $(x_1, x_2, x_3) =$

Date: 2015 May 1st .

AMS Subject Classifications: 35Q30 and 35B07.

$(r \cos \theta, r \sin \theta, x_3)$, axially symmetric solutions are of the form

$$v(x, t) = v_r(r, x_3, t)\vec{e}_r + v_\theta(r, x_3, t)\vec{e}_\theta + v_3(r, x_3, t)\vec{e}_3.$$

The components v_r, v_θ, v_3 are all independent of the angle of rotation θ . Here $\vec{e}_r, \vec{e}_\theta, \vec{e}_3$ are the basis vectors for \mathbb{R}^3 given by

$$\vec{e}_r = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right), \quad \vec{e}_\theta = \left(-\frac{x_2}{r}, \frac{x_1}{r}, 0\right), \quad \vec{e}_3 = (0, 0, 1).$$

It is known (see [5] for example) that v_r, v_3 and v_θ satisfy the equations

$$(1.2) \quad \begin{cases} \left(\Delta - \frac{1}{r^2}\right)v_r - (b \cdot \nabla)v_r + \frac{v_\theta^2}{r} - \partial_r p - \partial_t v_r = 0, \\ \left(\Delta - \frac{1}{r^2}\right)v_\theta - (b \cdot \nabla)v_\theta - \frac{v_\theta v_r}{r} - \partial_t v_\theta = 0, \\ \Delta v_3 - (b \cdot \nabla)v_3 - \partial_3 p - \partial_t v_3 = 0, \\ \frac{1}{r}\partial_r(rv_r) + \partial_3 v_3 = 0, \end{cases}$$

where $b(x, t) = (v_r, 0, v_3)$ and the last equation is the divergence-free condition. Here, Δ is the cylindrical, scalar Laplacian and ∇ is the cylindrical gradient field:

$$\Delta = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 + \partial_3^2, \quad \nabla = \left(\partial_r, \frac{1}{r}\partial_\theta, \partial_3\right).$$

Observe that the equation for v_θ does not depend on the pressure. Let $\Gamma = rv_\theta$, then

$$(1.3) \quad \Delta \Gamma - (b \cdot \nabla)\Gamma - \frac{2}{r}\partial_r \Gamma - \partial_t \Gamma = 0, \quad \operatorname{div} b = 0.$$

The vorticity $\omega = \operatorname{curl} v$ for axially symmetric solutions

$$\omega(x, t) = \omega_r \vec{e}_r + \omega_\theta \vec{e}_\theta + \omega_3 \vec{e}_3$$

is given by

$$(1.4) \quad \omega_r = -\partial_3 v_\theta, \quad \omega_\theta = \partial_3 v_r - \partial_r v_3, \quad \omega_3 = \partial_r v_\theta + \frac{v_\theta}{r}.$$

The equations of vorticity $\omega = \operatorname{curl} v$ in cylindrical form are (again, see [5] for example):

$$(1.5) \quad \begin{cases} \left(\Delta - \frac{1}{r^2}\right)\omega_r - (b \cdot \nabla)\omega_r + \omega_r \partial_r v_r + \omega_3 \partial_3 v_r - \partial_t \omega_r = 0, \\ \left(\Delta - \frac{1}{r^2}\right)\omega_\theta - (b \cdot \nabla)\omega_\theta + 2\frac{v_\theta}{r}\partial_3 v_\theta + \omega_\theta \frac{v_r}{r} - \partial_t \omega_\theta = 0, \\ \Delta \omega_3 - (b \cdot \nabla)\omega_3 + \omega_3 \partial_3 v_3 + \omega_r \partial_r v_3 - \partial_t \omega_3 = 0. \end{cases}$$

Although the axially symmetric Navier-Stokes equations is a special case of the full 3 dimensional one, our level of understanding had been roughly the same, with essential difficulty unresolved. One quick explanation of the difficulty goes as follows. Viewing (1.1) as a reaction diffusion equation. The standard theory for regularity requires the velocity to be bounded in suitable function space whose norm is invariant under standard scaling, such as $L^{p,q}$ with $\frac{3}{p} + \frac{2}{q} = 1$. However the only general a priori bound available is the energy estimate, which scales as $-1/2$. So there is a positive gap between the two which makes the equations supercritical.

Equation (1.2) has been studied by many authors in recent years. The following is a list which is far from complete. If the swirl $v_\theta = 0$, then long time ago, O. A. Ladyzhenskaya [11], M. R. Uchoviskii and B. I. Yudovich [20]), proved that finite energy solutions to (1.2)

are smooth for all time. See also the paper by S. Leonardi, J. Malek, J. Necas, and M. Pokorný [14]).

In the presence of swirl, it is not known in general if finite energy solutions blow up in finite time. However a lower bound for the possible blow up rate is known by the recent results of C.-C. Chen, R. M. Strain, T.-P. Tsai, and H.-T. Yau in [5], [6], G. Koch, N. Nadirashvili, G. Seregin, and V. Sverak in [10]. See also the work by G. Seregin and V. Sverak [18] for a localized version. These authors prove that if $|v(x, t)| \leq \frac{C}{r}$, then solutions are smooth for all time. Here C is any positive constant. Their result can be rephrased as: type I solutions are regular. See also the papers [12], [13] on further results in this direction. J. Neustupa and M. Pokorný [16] proved that the regularity of one component (either v_r or v_θ) implies regularity of the other components of the solution. See more refined results in [17] and the work of Ping Zhang and Ting Zhang [22]. Also proving regularity is the work of Q. Jiu and Z. Xin [9] under an assumption of sufficiently small zero-dimension scaled norms. D. Chae and J. Lee [4] also proved regularity results assuming finiteness of another certain zero-dimensional integral. G. Tian and Z. Xin [19] constructed a family of singular axially symmetric solutions with singular initial data. T. Hou and C. Li [7] found a special class of global smooth solutions. See also a recent extension: T. Hou, Z. Lei and C. Li [8].

Define

$$J = \frac{\omega_r}{r}, \quad \Omega = \frac{\omega_\theta}{r}.$$

Then the triple J, Ω, ω_3 satisfy the system

$$(1.6) \quad \begin{cases} \Delta J - (b \cdot \nabla)J + \frac{2}{r}\partial_r J + (\omega_r \partial_r + w_3 \partial_3) \frac{v_r}{r} - \partial_t J = 0, \\ \Delta \Omega - (b \cdot \nabla)\Omega + \frac{2}{r}\partial_r \Omega - \frac{2v_\theta}{r} J - \partial_t \Omega = 0, \\ \Delta w_3 - (b \cdot \nabla)w_3 + w_r \partial_r v_3 + w_3 \partial_3 v_3 - \partial_t w_3 = 0. \end{cases}$$

Here, in the second equation, we used the identity $rJ = w_r = -\partial_3 v_\theta$.

A great observation by Hui Chen, Daoyuan Fang and Ting Zhang in [2] is that the first two equations in (1.6) form a critical system under the standard scaling. Using this and a "magic formula" relating $\nabla(v_r/r)$ with w_θ/r by Changxing Miao and Xiaoxin Zheng [15], they obtained, among other things, an almost critical regularity condition on v_θ . For example it is proven that if $|v_\theta(x, t)| \leq C/r^{2-\epsilon}$ with $\epsilon > 0$, then solutions are regular.

In this paper we observe further that, all three equations are critical when viewed in a suitable way. Therefore the vorticity equation of 3 dimensional axially symmetric Navier-Stokes equations are critical instead of supercritical as commonly believed. This, together with a localization method in [21], allow us to prove Theorem 1.1 below, which provides a localized critical regularity condition on v_θ . It is tantalizing that our condition differs with the critical a priori bound ([4] or [16])

$$|v_\theta(x, t)| \leq \frac{C}{r}, \quad 0 < r \leq 1/2,$$

by the log factor $|\ln r|^{2+\epsilon}$ at worst. See the remarks below.

Now we introduce the function class where v_θ lives. It is defined in an integral way which is usually called the form boundedness condition, which is more general than the corresponding $L^{p,q}$ condition.

Definition 1.1. *We say the angular velocity v_θ is in the λ_1 critical class if there is a positive number $a < 1$ and another positive number λ_2 such that the inequality*

$$\int_0^t \int \left(\frac{|v_\theta|}{r} + v_\theta^2 \right) \psi^2 dy ds \leq \lambda_1 \int_0^t \int |\nabla \psi|^2 dy ds + \frac{\lambda_2}{a^2} \int_0^t \int \psi^2 dy ds$$

holds for all $t \geq 0$ and for all smooth $\psi = \psi(y, s)$, $s \in [0, t]$, satisfying the conditions (1) ψ is axially symmetric in y ; (2) $\psi(\cdot, s)$ is supported in the cylinder $D_{a,l} = \{(r, \theta, x_3) \mid 0 \leq r < a, -l < x_3 < l, 0 \leq \theta < 2\pi\}$ for some $l \geq a$.

Remark 1.1. Clearly the class is scaling invariant. A function v_θ is in the λ_1 critical class for all $\lambda_1 > 0$ if it satisfies $|v_\theta(x, t)| \leq \frac{C}{r|\ln r|^{2+\epsilon}}$, $r < 1/2$. Here $C > 0$, $\epsilon > 0$ are arbitrary positive constant. This claim will be proven at the end of the paper. One may also take $\epsilon = 0$ but replace r by r/a and C by a small constant in the bound, by virtue of the 2 dimensional Hardy's inequality.

Here is the main result of the paper.

Theorem 1.1. *Let v be a Leray-Hopf axially symmetric solution of the three-dimensional Navier-Stokes equations in $\mathbb{R}^3 \times (0, \infty)$ with initial data $v_0 = v(\cdot, 0) \in L^2(\mathbb{R}^3)$. Assume further $rv_{0,\theta} \in L^\infty(\mathbb{R}^3)$.*

There exists a positive number λ_1 . Suppose v_θ is in the λ_1 critical class. Then v is smooth for all time.

Remark 1.2. The size of λ_1 is estimated in (2.36). It is an absolute constant depending on the L^2 norm of the Riesz operators. There is no size restriction on λ_2 . Also the a^2 in the definition can be replaced by any positive continuous function of a . But this may break the scaling invariance.

The theorem will be proven in the next section. The following are some notations to be frequently used. We use $x = (x_1, x_2, x_3)$ to denote a point in \mathbb{R}^3 for rectangular coordinates, and in the cylindrical system we use $r = \sqrt{x_1^2 + x_2^2}$, $\theta = \tan^{-1} \frac{x_2}{x_1}$. We will use $S(v_0, \dots), C(v_0, \dots)$ to denote positive constants which depend on the initial velocity v_0 etc. Also C denotes absolute constant which may change value.

Let us explain why the vortex stretching terms in (1.6) are critical. For example the term $w_3 \partial_3 v_3$ where $\partial_3 v_3$ being viewed as a potential of the unknown function w_3 is certainly supercritical. However, we view $w_3 = \partial_r v_\theta + \frac{v_\theta}{r}$ as the potential and $\partial_3 v_3$ as the unknown. Since it is known that $|v_\theta| \leq C/r$, we see that w_3 now scales as -2 power of the distance. This scaling shows w_3 is a critical potential function. The unknown function $\partial_3 v_3$ scales the same way as the vorticity w . By exploiting the integral relations between v and w , we can convert $\partial_3 v_3$ into w_r, w_3, w_θ . This, combined with the observation [2] about the first two equations in (1.6), imply that all the vortex stretching terms are critical. Next we carry a local energy estimate for (J, Ω, w_z) via equations (1.6). Once we know the potential terms are critical, the drift terms can be treated by an old small trick in [21], the proof thus goes through.

2. PROOF OF THE THEOREM

The proof is divided into several steps. We may assume that v is smooth up to a given time t .

Step 1. Choose suitable test functions for equations (1.6).

It is well known that singularity can possibly appear only on a finite segment of the x_3 axis ([3] for suitable solutions and [1] for general ones). So by picking any positive number $a \leq 1$ and another positive number $l > a$, which may depend on the initial velocity v_0 , we can ensure that v is regular outside of the domain $D_1 = \{(r, \theta, x_3) \mid 0 \leq r < a/2, -l/2 < x_3 < l/2, 0 \leq \theta < 2\pi\}$ for all time. Let $\phi = \phi(r, x_3)$ be a axially symmetric cut off function in $D_2 = \{(r, \theta, x_3) \mid 0 \leq r < a, -l < x_3 < l, 0 \leq \theta < 2\pi\}$ such that $\phi = 1$ on $D_3 = \{(r, \theta, x_3) \mid 0 \leq r < 2a/3, -2l/3 < x_3 < 2l/3, 0 \leq \theta < 2\pi\}$ and $\phi = 0$ on D_2^c and also $\frac{|\nabla \phi|}{\phi^{1/2}} \leq C/a$, $|\nabla^2 \phi| \leq C/a^2$.

Use $J\phi^2$, $\Omega\phi^2$ and $w_3\phi^2$ as test functions in equations 1, 2 and 3 in (1.6) respectively. After integration on the region $D_2 \times [0, t]$ for $t > 0$ we find that

$$\begin{aligned}
 (2.1) \quad L_1 &\equiv - \int_0^t \int \Delta J J \phi^2 dy ds - \int_0^t \int \frac{2}{r} \partial_r J J \phi^2 dy ds + \int_0^t \int \partial_t J J \phi^2 dy ds \\
 &= - \int_0^t \int b \nabla J J \phi^2 dy ds + \int_0^t \int (w_r \partial_r \frac{v_r}{r} + w_3 \partial_3 \frac{v_r}{r}) J \phi^2 dy ds \\
 &\equiv R_1 + T_1.
 \end{aligned}$$

$$\begin{aligned}
 (2.2) \quad L_2 &\equiv - \int_0^t \int \Delta \Omega \Omega \phi^2 dy ds - \int_0^t \int \frac{2}{r} \partial_r \Omega \Omega \phi^2 dy ds + \int_0^t \int \partial_t \Omega \Omega \phi^2 dy ds \\
 &= - \int_0^t \int b \nabla \Omega \Omega \phi^2 dy ds - \int_0^t \int \frac{2v_\theta}{r} J \Omega \phi^2 dy ds \\
 &\equiv R_2 + T_2.
 \end{aligned}$$

$$\begin{aligned}
 (2.3) \quad L_3 &\equiv - \int_0^t \int \Delta w_3 w_3 \phi^2 dy ds + \int_0^t \int \partial_t w_3 w_3 \phi^2 dy ds \\
 &= - \int_0^t \int b \nabla w_3 w_3 \phi^2 dy ds + \int_0^t \int (w_3 \partial_3 v_3 + w_r \partial_r v_3) w_3 \phi^2 dy ds \\
 &\equiv R_3 + T_3.
 \end{aligned}$$

The left hand side of the three equalities L_1 , L_2 and L_3 can be treated by routine integration by parts which shows:

$$\begin{aligned}
 L_1 &= \int_0^t \int |\nabla J|^2 \phi^2 dy ds + \int_0^t \int J^2(0, y_3, t) \phi^2 dy_3 dr dt + \frac{1}{2} \int J^2 \phi^2 dy \Big|_0^t \\
 &\quad - \int_0^t \int \nabla J J \nabla \phi^2 dy ds + \int_0^t \int J^2 \frac{\partial_r \phi^2}{r} dy ds.
 \end{aligned}$$

Therefore

$$L_1 \geq \frac{1}{2} \int_0^t \int |\nabla J|^2 \phi^2 dy ds + \frac{1}{2} \int J^2 \phi^2 dy \Big|_0^t - 2 \int_0^t \int J^2 |\nabla \phi|^2 dy ds + \int_0^t \int J^2 \frac{\partial_r \phi^2}{r} dy ds.$$

By our choice of the cut off function ϕ , we know v is regular in the supports of $\nabla \phi$ and $\partial_r \phi$, which is bounded away from the singular set by a distance $a/6$. So there is a positive constant $S = S(v_0, a, l)$ such that

$$(2.4) \quad L_1 \geq \frac{1}{2} \int_0^t \int |\nabla J|^2 \phi^2 dy ds + \frac{1}{2} \int J^2 \phi^2 dy \Big|_0^t - CtS(v_0, a, l).$$

Here we recall that J and Ω are all smooth functions if v is smooth. Similarly

$$(2.5) \quad L_2 \geq \frac{1}{2} \int_0^t \int |\nabla \Omega|^2 \phi^2 dy ds + \frac{1}{2} \int \Omega^2 \phi^2 dy \Big|_0^t - CtS(v_0, a, l),$$

$$(2.6) \quad L_3 \geq \frac{1}{2} \int_0^t \int |\nabla w_3|^2 \phi^2 dy ds + \frac{1}{2} \int w_3^2 \phi^2 dy \Big|_0^t - CtS(v_0, a, l).$$

We remark that $S(v_0, a, l)$ may blow up when $a \rightarrow 0$. But we will make a small and fixed.

Substituting (2.4), (2.5) and (2.6) into (2.1), (2.2) and (2.3) respectively, we deduce

$$(2.7) \quad \begin{aligned} & \int (J^2 + \Omega^2 + w_3^2) \phi^2 dy \Big|_0^t + \int_0^t \int (|\nabla J|^2 + |\nabla \Omega|^2 + |\nabla w_3|^2) \phi^2 dy ds \\ & \leq 2(R_1 + R_2 + R_3) + 2(T_1 + T_2 + T_3) + CS(v_0, a, l). \end{aligned}$$

We are going to bound the right hand side in the next few steps.

Step 2. bounds on $R_1 + R_2 + R_3$, the drift terms.

These terms are generated by $b = v_r \vec{e}_r + v_3 \vec{e}_3$ which is supercritical. However since these are given by divergence free drift terms, they can be bounded as done in [21]. We present a proof for completeness.

Since $\operatorname{div} b = 0$, we have

$$\begin{aligned} R_1 &= - \int_0^t \int b \cdot (\nabla J)(J\phi^2) dy ds \\ &= \int_0^t \int b \cdot (\nabla \phi) \phi J^2 dy ds \\ &\leq \left| \int \left(b \phi^{3/2} |J|^{3/2} \right) \left(\frac{\nabla \phi}{\phi^{1/2}} |J|^{1/2} \right) dy ds \right|. \end{aligned}$$

By Hölder's inequality with exponents $\frac{4}{3}$ and 4,

$$R_1 \leq \left(\int_0^t \int |b|^{\frac{4}{3}} \left(\phi^{3/2} |J|^{3/2} \right)^{\frac{4}{3}} dy ds \right)^{\frac{3}{4}} \left(\int_0^t \int \left(\frac{|\nabla \phi|}{\phi^{1/2}} |J|^{1/2} \right)^4 dy ds \right)^{\frac{1}{4}}.$$

Using properties of the cutoff function we find:

$$R_1 \leq \left(\int_0^t \int |b|^{\frac{4}{3}} (J\phi)^2 dy ds \right)^{\frac{3}{4}} \frac{C}{a} \left(\int_0^t \int_{\text{supp}|\nabla\phi|} J^2 dy ds \right)^{\frac{1}{4}}.$$

Next we fix $\epsilon_1 > 0$ and we apply Young's inequality, with exponents $\frac{4}{3}$ and 4:

$$\begin{aligned} R_1 &\leq \left(\frac{4}{3} \epsilon_1 \right)^{\frac{3}{4}} \left(\int_0^t \int |b|^{\frac{4}{3}} (J\phi)^2 dy ds \right)^{\frac{3}{4}} \cdot \left(\frac{4}{3} \epsilon_1 \right)^{-\frac{3}{4}} \frac{C}{a} \left(\int_0^t \int_{\text{supp}|\nabla\phi|} J^2 dy ds \right)^{\frac{1}{4}} \\ &\leq \epsilon_1 \int_0^t \int |b|^{\frac{4}{3}} (J\phi)^2 dy ds + \frac{C\epsilon_1^{-3}}{a^4} \int_0^t \int_{\text{supp}|\nabla\phi|} J^2 dy ds. \end{aligned}$$

Thus,

$$(2.8) \quad |R_1| \leq \epsilon_1 c_0 \|b\|_{2,\infty}^{4/3} \int_0^t \int |\nabla(J\phi)|^2 dy ds + \frac{C\epsilon_1^{-3}}{a^4} \int_0^t \int_{\text{supp}|\nabla\phi|} J^2 dy ds.$$

This last inequality holds as a result of the standard energy estimate, Hölder's inequality with exponents $\frac{3}{2}$ and 3, and the 3 dimensional Sobolev Inequality,

$$\begin{aligned} \int_0^t \int |b|^{\frac{4}{3}} (J\phi)^2 dy ds &\leq \int_0^t \left(\int |b|^2 dy \right)^{\frac{2}{3}} \left(\int (J\phi)^6 dy \right)^{\frac{1}{3}} ds \\ &\leq c_0 \|b\|_{2,\infty}^{4/3} \int_0^t \int |\nabla(J\phi)|^2 dy ds. \end{aligned}$$

By choosing ϵ_1 suitably, we deduce

$$(2.9) \quad |R_1| \leq \frac{1}{8} \int_0^t \int |\nabla J|^2 \phi^2 dy ds + CS(v_0, a, l),$$

where we have used the fact that v is regular in the support of $\nabla\phi$ for all time. In exactly the same manner, we find that

$$(2.10) \quad |R_1| + |R_2| + |R_3| \leq \frac{1}{8} \int_0^t \int (|\nabla J|^2 + |\nabla\Omega|^2 + |\nabla w_3|^2) \phi^2 dy ds + CS(v_0, a, l),$$

Step 3. bounds on T_1 and T_2 .

In this step we follow the idea in [CFZ] with one modification, namely a localized version of a formula of Miao and Zheng which relates $\frac{v_\tau}{r}$ with $\frac{w_\theta}{r}$. The rest of the step is divided into a few sub steps.

step 3.1

First we work on the easy one T_2 defined in (2.2).

$$\begin{aligned} T_2 &= - \int_0^t \int \frac{2v_\theta}{r} J\Omega\phi^2 dy ds \\ &\leq \int_0^t \int \frac{|v_\theta|}{r} (J\phi)^2 dy ds + \int_0^t \int \frac{|v_\theta|}{r} (\Omega\phi)^2 dy ds. \end{aligned}$$

By our assumption on v_θ , this implies

$$T_2 \leq \lambda_1 \int_0^t \int (|\nabla(J\phi)|^2 + |\nabla(\Omega\phi)|^2) dy ds + \lambda_2 \int_0^t \int [(J\phi)^2 + (\Omega\phi)^2] dy ds.$$

Let us write $\nabla(J\phi) = \nabla J\phi + J\nabla\phi$. As mentioned earlier, J is regular in the support of $\nabla\phi$. Hence

$$(2.11) \quad T_2 \leq 2\lambda_1 \int_0^t \int (|\nabla J|^2 + |\nabla\Omega|^2) \phi^2 dy ds + \lambda_2 \int_0^t \int [(J\phi)^2 + (\Omega\phi)^2] dy ds + CtS(v_0, a, l).$$

Here we also did the same argument for $\nabla(\Omega\phi)$.

step 3.2

Next we turn to T_1 . From (2.1),

$$\frac{dT_1}{dt} = \int (w_r \partial_r \frac{v_r}{r} + w_3 \partial_3 \frac{v_r}{r}) J \phi^2 dy$$

Using the relation $w_r = -\partial_3 v_\theta$, $w_3 = \frac{1}{r} \partial_r (rv_\theta)$ and integration by parts, we see that

$$\begin{aligned} \frac{dT_1}{dt} &= - \int \partial_3 v_\theta \partial_r \left(\frac{v_r}{r} \right) J \phi^2 dy + \int \frac{1}{r} \partial_r (rv_\theta) \partial_3 \left(\frac{v_r}{r} \right) J \phi^2 dy \\ &= \int v_\theta \partial_3 \partial_r \left(\frac{v_r}{r} \right) J \phi^2 dy + \int v_\theta \partial_r \left(\frac{v_r}{r} \right) \partial_3 (J \phi^2) dy \\ &\quad - \int v_\theta \partial_r \partial_3 \left(\frac{v_r}{r} \right) J \phi^2 dy - \int v_\theta \partial_3 \left(\frac{v_r}{r} \right) \partial_r (J \phi^2) dy. \end{aligned}$$

Notice that the first and third term on the right hand side of the last equality cancel. Therefore, we deduce

$$\begin{aligned} \frac{dT_1}{dt} &= \int v_\theta \partial_r \left(\frac{v_r}{r} \right) (\partial_3 J) \phi^2 dy - \int v_\theta \partial_3 \left(\frac{v_r}{r} \right) (\partial_r J) \phi^2 dy \\ &\quad + \int v_\theta \partial_r \left(\frac{v_r}{r} \right) J \partial_r \phi^2 dy - \int v_\theta \partial_3 \left(\frac{v_r}{r} \right) J \partial_r \phi^2 dy. \end{aligned}$$

This implies, since the last two terms in the above identity are bounded, that

$$\begin{aligned} T_1 &\leq \frac{1}{8} \int_0^t \int |\partial_3 J|^2 \phi^2 dy + 2 \int_0^t \int v_\theta^2 \left| \partial_r \frac{v_r}{r} \right|^2 \phi^2 dy \\ &\quad + \frac{1}{8} \int_0^t \int |\partial_r J|^2 \phi^2 dy + 2 \int_0^t \int v_\theta^2 \left| \partial_3 \frac{v_r}{r} \right|^2 \phi^2 dy + CtS(v_0, a, l). \end{aligned}$$

By our condition on v_θ again, we find that

$$\begin{aligned} T_1 &\leq \frac{1}{8} \int_0^t \int |\nabla J|^2 \phi^2 dy + CtS(v_0, a, l) + 2\lambda_1 \int_0^t \int |\nabla(\phi \partial_r \frac{v_r}{r})|^2 dy + 2\lambda_2 \int_0^t \int (\phi \partial_r \frac{v_r}{r})^2 dy \\ &\quad + 2\lambda_1 \int_0^t \int |\nabla(\phi \partial_3 \frac{v_r}{r})|^2 dy + 2\lambda_2 \int_0^t \int (\phi \partial_3 \frac{v_r}{r})^2 dy. \end{aligned}$$

This implies, after using again the fact that v is smooth in the support of $\nabla\phi$, that

$$(2.12) \quad \begin{aligned} T_1 \leq & \frac{1}{8} \int_0^t \int |\nabla J|^2 \phi^2 dy + CtS(v_0, a, l) + 4\lambda_1 \int_0^t \int |\nabla(\partial_r(\phi \frac{v_r}{r}))|^2 dy + 4\lambda_2 \int_0^t \int (\partial_r(\phi \frac{v_r}{r}))^2 dy \\ & + 4\lambda_1 \int_0^t \int |\nabla(\partial_3(\phi \frac{v_r}{r}))|^2 dy + 4\lambda_2 \int_0^t \int (\partial_3(\phi \frac{v_r}{r}))^2 dy. \end{aligned}$$

Here the constant C may have changed. We need to bound the last 4 terms on the preceding inequality. For this purpose, we first need to prove the following localized version of a nice identity by Miao and Zheng. For any $q \in (1, \infty)$, there is a positive constant c_q such that

$$(2.13) \quad \begin{aligned} \|\nabla(\phi \partial_r \frac{v_r}{r})\|_q &\leq c_q \|\Omega\phi\|_q + S(v_0, a, l), \\ \|\nabla^2(\phi \partial_r \frac{v_r}{r})\|_q &\leq c_q \|\nabla(\Omega\phi)\|_q + S(v_0, a, l). \end{aligned}$$

Here, as always $\Omega = w_\theta/r$. The proof of theses inequalities is given in

step 3.3. From the identity

$$\Delta b = -\nabla \times (w_\theta \vec{e}_\theta) = \left(\partial_3(w_\theta \frac{x_1}{r}), \partial_3(w_\theta \frac{x_2}{r}), \partial_1(w_\theta \frac{x_1}{r}) - \partial_2(w_\theta \frac{x_2}{r}) \right),$$

and $b = v_r(\frac{x_1}{r}, \frac{x_2}{r}, 0) + v_3(0, 0, 1)$, we see that

$$(2.14) \quad \Delta(v_r \frac{x_1}{r}) = \partial_3(x_1 \Omega), \quad \Delta(v_r \frac{x_2}{r}) = \partial_3(x_2 \Omega).$$

Therefore

$$(2.15) \quad \Delta(v_r \frac{x_1}{r} \phi) = \partial_3(x_1 \Omega \phi) - x_1 \Omega \partial_3 \phi + 2\nabla(v_r \frac{x_1}{r}) \nabla \phi + v_r \frac{x_1}{r} \Delta \phi.$$

Likewise

$$(2.16) \quad \Delta(v_r \frac{x_2}{r} \phi) = \partial_3(x_2 \Omega \phi) - x_2 \Omega \partial_3 \phi + 2\nabla(v_r \frac{x_2}{r}) \nabla \phi + v_r \frac{x_2}{r} \Delta \phi.$$

Inverting the Laplace operator, we infer

$$(2.17) \quad v_r \frac{x_1}{r} \phi = \Delta^{-1} \partial_3(x_1 \Omega \phi) - \Delta^{-1} [x_1 \Omega \partial_3 \phi - 2\nabla(v_r \frac{x_1}{r}) \nabla \phi - v_r \frac{x_1}{r} \Delta \phi],$$

$$(2.18) \quad v_r \frac{x_2}{r} \phi = \Delta^{-1} \partial_3(x_2 \Omega \phi) - \Delta^{-1} [x_2 \Omega \partial_3 \phi - 2\nabla(v_r \frac{x_2}{r}) \nabla \phi - v_r \frac{x_2}{r} \Delta \phi].$$

Multiplying (2.17) by x_1 , (2.18) by x_2 and taking the sum, we arrive at

$$(2.19) \quad v_r \phi = \sum_{i=1}^2 \frac{x_i}{r} \Delta^{-1} \partial_3(x_i \Omega \phi) - \sum_{i=1}^2 \frac{x_i}{r} \Delta^{-1} [x_i \Omega \partial_3 \phi - 2\nabla(v_r \frac{x_i}{r}) \nabla \phi - v_r \frac{x_i}{r} \Delta \phi].$$

Since ϕ is axially symmetric and $x_1/r = \cos \theta$, $x_2/r = \sin \theta$, we can write, for $i = 1, 2$, that

$$\nabla(v_r \frac{x_i}{r}) \nabla \phi = \frac{x_i}{r} (\partial_r v_r \partial_r \phi + \partial_3 v_r \partial_3 \phi).$$

This turns (2.19) into

$$(2.20) \quad \begin{aligned} v_r \phi &= \sum_{i=1}^2 \frac{x_i}{r} \Delta^{-1} \partial_3 (x_i \Omega \phi) - \sum_{i=1}^2 \frac{x_i}{r} \Delta^{-1} (x_i f), \\ f &\equiv \Omega \partial_3 \phi - 2 \frac{\partial_r v_r}{r} \partial_r \phi - 2 \frac{\partial_3 v_r}{r} \partial_3 \phi - \frac{v_r}{r} \Delta \phi. \end{aligned}$$

Note the function f is compactly supported, axially symmetric and point-wise bounded, due to the choice of the cut off function ϕ .

According to [15], the following operator identity holds, at east when acting on compactly supported functions,

$$(2.21) \quad \sum_{i=1}^2 \frac{x_i}{r} \Delta^{-1} x_i = r \Delta^{-1} - 2 \partial_r \Delta^{-2}.$$

Since their proof is very sharp and cute, we repeat it here for completeness. Notice that

$$\sum_{i=1}^2 x_i [x_i, \Delta^{-1}] = \sum_{i=1}^2 x_i^2 \Delta^{-1} - \sum_{i=1}^2 x_i \Delta^{-1} x_i = r^2 \Delta^{-1} - \sum_{i=1}^2 x_i \Delta^{-1} x_i.$$

Hence

$$(2.22) \quad \sum_{i=1}^2 \frac{x_i}{r} \Delta^{-1} x_i = r \Delta^{-1} - \sum_{i=1}^2 \frac{x_i}{r} [x_i, \Delta^{-1}].$$

On the other hand

$$\Delta [x_i, \Delta^{-1}] = \Delta (x_i \Delta^{-1}) - \Delta \Delta^{-1} x_i = 2 \partial_i \Delta^{-1},$$

which implies

$$[x_i, \Delta^{-1}] = 2 \partial_i \Delta^{-2}.$$

Substituting this to the last term in (2.22), one obtains (2.21). Plugging (2.21) into the first identity in (2.20), we find that

$$(2.23) \quad \frac{v_r}{r} \phi = (\Delta^{-1} \partial_3 - 2 \frac{\partial_r}{r} \Delta^{-2} \partial_3) (\Omega \phi) - (\Delta^{-1} - 2 \frac{\partial_r}{r} \Delta^{-2}) f.$$

Recall that both $\Omega \phi$ and f are axially symmetric. When the operator $\frac{\partial_r}{r}$ acts on these functions, it can be written as

$$\frac{\partial_r}{r} = \Delta - \partial_r^2 - \partial_3^2.$$

Plugging this into (2.23), we deduce

$$(2.24) \quad \nabla \left(\frac{v_r}{r} \phi \right) = \Pi_1 (\Omega \phi) + \Pi_0 f,$$

where Π_1 and $\nabla \Pi_0$ are Riesz type singular integral operators that map L^q to L^q , $q \in (1, \infty)$ and Π_0 is a smoothing integral operator. Since f is bounded and compactly supported, this proves (2.13). We have used the fact that the gradient ∇ does not involve the derivative in \vec{e}_θ direction, when acting on axially symmetric functions.

step 3.4.

Now we can take $q = 2$ in (2.13) and substitute it to (2.12) to obtain

$$(2.25) \quad \begin{aligned} T_1 \leq & \frac{1}{8} \int_0^t \int |\nabla J|^2 \phi^2 dy + CtS(v_0, a, l) + 4\lambda_1 c_2 \int_0^t \int |\nabla(\Omega\phi)|^2 dy + 4\lambda_2 c_2 \int_0^t \int (\Omega\phi)^2 dy \\ & + 4\lambda_1 c_2 \int_0^t \int |\nabla(\Omega\phi)|^2 dy + 4\lambda_2 c_2 \int_0^t \int (\Omega\phi)^2 dy. \end{aligned}$$

This, together with (2.11), yield

$$(2.26) \quad \begin{aligned} T_1 + T_2 \leq & \left(\frac{1}{8} + 2\lambda_1 + 9\lambda_1 c_2\right) \int_0^t \int (|\nabla J|^2 + |\nabla\Omega|^2) \phi^2 dy ds \\ & + (\lambda_2 + 8\lambda_2 c_2) \int_0^t \int [(J\phi)^2 + (\Omega\phi)^2] dy ds + CtS(v_0, a, l). \end{aligned}$$

In the above we have used the product formula $(\nabla\Omega)\phi = \nabla(\Omega\phi) - \Omega\nabla\phi$. This completes Step 3.

Step 4. bounds on T_3 .

Using $w_3 = \frac{1}{r}\partial_r(rv_\theta)$, we compute

$$\begin{aligned} \int w_3 \partial_3 v_3 w_3 \phi^2 dy &= \int \int_0^\infty \partial_r(rv_\theta) \partial_3 v_3 w_3 \phi^2 dr dy_3 \\ &= - \int \int_0^\infty rv_\theta \partial_r \partial_3 v_3 w_3 \phi^2 dr dy_3 - \int \int_0^\infty rv_\theta \partial_3 v_3 \partial_r w_3 \phi^2 dr dy_3 - \int \int_0^\infty rv_\theta \partial_3 v_3 w_3 \partial_r \phi^2 dr dy_3 \\ &= - \int v_\theta \partial_r \partial_3 v_3 w_3 \phi^2 dy - \int v_\theta \partial_3 v_3 \partial_r w_3 \phi^2 dy - \int v_\theta \partial_3 v_3 w_3 \partial_r \phi^2 dy. \end{aligned}$$

Next, using $w_r = -\partial_3 v_\theta$, we have

$$\begin{aligned} \int w_r \partial_r v_3 w_3 \phi^2 dy &= - \int \partial_3 v_\theta \partial_r v_3 w_3 \phi^2 dy \\ &= \int v_\theta \partial_3 \partial_r v_3 w_3 \phi^2 dy + \int v_\theta \partial_r v_3 \partial_3 w_3 \phi^2 dy + \int v_\theta \partial_r v_3 w_3 \partial_3 \phi^2 dy. \end{aligned}$$

Adding the previous two equalities and noting that the first terms on the right hand sides cancel, we obtain

$$\begin{aligned} T_3 = & - \int_0^t \int v_\theta \partial_3 v_3 \partial_r w_3 \phi^2 dy ds - \int_0^t \int v_\theta \partial_3 v_3 w_3 \partial_r \phi^2 dy ds \\ & + \int_0^t \int v_\theta \partial_r v_3 \partial_3 w_3 \phi^2 dy ds + \int_0^t \int v_\theta \partial_r v_3 w_3 \partial_3 \phi^2 dy ds. \end{aligned}$$

As before, all terms involving derivatives of ϕ are bounded by $CtS(v_0, a, l)$. Thus

$$(2.27) \quad \begin{aligned} T_3 \leq & - \int_0^t \int v_\theta \partial_3 v_3 \partial_r w_3 \phi^2 dy ds + \int_0^t \int v_\theta \partial_r v_3 \partial_3 w_3 \phi^2 dy ds + CtS(v_0, a, l) \\ \equiv & I_1 + I_2 + CtS(v_0, a, l). \end{aligned}$$

We will bound I_1 first. By our condition on v_θ ,

$$\begin{aligned} I_1 &\leq \frac{1}{8} \int_0^t \int |\partial_r w_3|^2 \phi^2 dy ds + 2 \int_0^t \int v_\theta^2 |\partial_3 v_3|^2 \phi^2 dy ds \\ &\leq \frac{1}{8} \int_0^t \int |\partial_r w_3|^2 \phi^2 dy ds + 2\lambda_1 \int_0^t \int |\nabla(\phi \partial_3 v_3)|^2 dy ds + 2\lambda_2 \int_0^t \int |\partial_3 v_3|^2 \phi^2 dy ds. \end{aligned}$$

Consequently

$$(2.28) \quad I_1 \leq \frac{1}{8} \int_0^t \int |\partial_r w_3|^2 \phi^2 dy ds + 3\lambda_1 \int_0^t \int |\nabla \partial_3 v_3|^2 \phi^2 dy ds + CtS(v_0, a, l, \lambda_2).$$

We need to bound the second term on the right hand side. To this end we call the relation for the full three dimensional velocity and vorticity:

$$-\Delta \partial_i v = \nabla \times \partial_i w,$$

where $i = 1, 2, 3$. Using $\partial_i v \phi^2$ as a test function and integrate, we know that

$$\begin{aligned} &\int |\nabla \partial_i v|^2 \phi^2 dy + \int \partial_j \partial_i v \partial_i v \partial_j \phi^2 dy = \int (\nabla \times \partial_i w) \partial_i v \phi^2 dy \\ &= - \int (\nabla \times w) \partial_i \partial_i v \phi^2 dy - \int (\nabla \times w) \partial_i v \partial_i \phi^2 dy \\ &\leq \frac{1}{2} \int |\nabla \partial_i v|^2 \phi^2 dy + \frac{1}{2} \int |\nabla \times w|^2 \phi^2 dy - \int (\nabla \times w) \partial_i v \partial_i \phi^2 dy. \end{aligned}$$

Since the terms involving derivatives of ϕ are bounded, this shows

$$\begin{aligned} (2.29) \quad &\int_0^t \int |\nabla \partial_3 v_3|^2 \phi^2 dy ds \leq \int_0^t \int |\nabla \times w|^2 \phi^2 dy ds + CtS(v_0, a, l) \\ &\leq \int_0^t \int |\nabla w|^2 \phi^2 dy ds + CtS(v_0, a, l), \end{aligned}$$

and

$$\begin{aligned} (2.30) \quad &\int_0^t \int |\nabla \partial_r v_3|^2 \phi^2 dy ds \leq \int_0^t \int |\nabla \times w|^2 \phi^2 dy ds + CtS(v_0, a, l) \\ &\leq \int_0^t \int |\nabla w|^2 \phi^2 dy ds + CtS(v_0, a, l). \end{aligned}$$

Here the constant C may have changed when we drop the cross product, which can be done through integration by parts that produces extra bounded terms involving $\nabla \phi$.

Substituting (2.29) into the second term on the right hand side of (2.28), we reach

$$(2.31) \quad I_1 \leq \frac{1}{8} \int_0^t \int |\partial_r w_3|^2 \phi^2 dy ds + 3\lambda_1 \int_0^t \int |\nabla w|^2 \phi^2 dy ds + CtS(v_0, a, l, \lambda_1, \lambda_2).$$

Similarly, by our condition on v_θ ,

$$\begin{aligned} I_2 &\leq \frac{1}{8} \int_0^t \int |\partial_3 w_3|^2 \phi^2 dy ds + 2 \int_0^t \int v_\theta^2 |\partial_r v_3|^2 \phi^2 dy ds \\ &\leq \frac{1}{8} \int_0^t \int |\partial_3 w_3|^2 \phi^2 dy ds + 2\lambda_1 \int_0^t \int |\nabla(\phi \partial_r v_3)|^2 dy ds + 2\lambda_2 \int_0^t \int |\partial_r v_3|^2 \phi^2 dy ds. \end{aligned}$$

This with (2.30) imply that

$$(2.32) \quad I_2 \leq \frac{1}{8} \int_0^t \int |\partial_3 w_3|^2 \phi^2 dy ds + 3\lambda_1 \int_0^t \int |\nabla w|^2 \phi^2 dy ds + CtS(v_0, a, l, \lambda_1, \lambda_2).$$

Substituting (2.31) and (2.32) into (2.27), we deduce the bound for T_3 , i.e.

$$(2.33) \quad T_3 \leq \frac{1}{8} \int_0^t \int |\nabla w_3|^2 \phi^2 dy ds + 6\lambda_1 \int_0^t \int |\nabla w|^2 \phi^2 dy ds + CtS(v_0, a, l, \lambda_1, \lambda_2).$$

Step 5. conclusion of the proof.

Combining (2.26) with (2.33), we get

$$\begin{aligned} (2.34) \quad T_1 + T_2 + T_3 &\leq \left(\frac{1}{8} + 2\lambda_1 + 9\lambda_1 c_2\right) \int_0^t \int (|\nabla J|^2 + |\nabla \Omega|^2) \phi^2 dy ds \\ &\quad + (\lambda_2 + 8\lambda_2 c_2) \int_0^t \int [(J\phi)^2 + (\Omega\phi)^2] dy ds + \frac{1}{8} \int_0^t \int |\nabla w_3|^2 \phi^2 dy ds \\ &\quad + 6\lambda_1 \int_0^t \int |\nabla w|^2 \phi^2 dy ds + CtS(v_0, a, l, \lambda_1, \lambda_2). \end{aligned}$$

This, (2.10) and (2.7) together give

$$\begin{aligned} &\int (J^2 + \Omega^2 + w_3^2) \phi^2 dy \Big|_0^t + \int_0^t \int (|\nabla J|^2 + |\nabla \Omega|^2 + |\nabla w_3|^2) \phi^2 dy ds \\ &\leq \frac{1}{4} \int_0^t \int (|\nabla J|^2 + |\nabla \Omega|^2 + |\nabla w_3|^2) \phi^2 dy ds \\ &\quad \left(\frac{1}{4} + 4\lambda_1 + 18\lambda_1 c_2\right) \int_0^t \int (|\nabla J|^2 + |\nabla \Omega|^2) \phi^2 dy ds \\ &\quad + 2(\lambda_2 + 8\lambda_2 c_2) \int_0^t \int [(J\phi)^2 + (\Omega\phi)^2] dy ds + \frac{1}{4} \int_0^t \int |\nabla w_3|^2 \phi^2 dy ds \\ &\quad + 12\lambda_1 \int_0^t \int |\nabla w|^2 \phi^2 dy ds + CtS(v_0, a, l, \lambda_1, \lambda_2). \end{aligned}$$

Hence

$$\begin{aligned}
(2.35) \quad & \int (J^2 + \Omega^2 + w_3^2) \phi^2 dy \Big|_0^t + \frac{1}{4} \int_0^t \int (|\nabla J|^2 + |\nabla \Omega|^2 + |\nabla w_3|^2) \phi^2 dy ds \\
& \leq (4 + 18c_2) \lambda_1 \int_0^t \int (|\nabla J|^2 + |\nabla \Omega|^2) \phi^2 dy ds + 12\lambda_1 \int_0^t \int |\nabla w|^2 \phi^2 dy ds \\
& \quad + 2\lambda_2(1 + 8c_2) \int_0^t \int [(J\phi)^2 + (\Omega\phi)^2] dy ds + CtS(v_0, a, l, \lambda_1, \lambda_2).
\end{aligned}$$

There is still a little work to do, namely to bound the second term on the right hand side by the left hand side. Notice that w is axially symmetric. Hence

$$\begin{aligned}
|\nabla w|^2 &= |\partial_r w_r|^2 + |\partial_r w_\theta|^2 + |\partial_3 w_r|^2 + |\partial_3 w_\theta|^2 + |\nabla w_3|^2 \\
&= |\partial_r(Jr)|^2 + |\partial_r(\Omega r)|^2 + r^2 |\partial_3 J|^2 + r^2 |\partial_3 \Omega|^2 + |\nabla w_3|^2 \\
&= |r \partial_r J + J|^2 + |r \partial_r \Omega + \Omega|^2 + r^2 |\partial_3 J|^2 + r^2 |\partial_3 \Omega|^2 + |\nabla w_3|^2 \\
&\leq 2r^2 |\partial_r J|^2 + 2J^2 + 2r^2 |\partial_r \Omega|^2 + 2\Omega^2 + r^2 |\partial_3 J|^2 + r^2 |\partial_3 \Omega|^2 + |\nabla w_3|^2.
\end{aligned}$$

Hence

$$|\nabla w|^2 \leq 2r^2 (|\nabla J|^2 + |\nabla \Omega|^2) + |\nabla w_3|^2 + 2(J^2 + \Omega^2).$$

Plugging this to the second term on the right hand side of (2.35), we arrive at

$$\begin{aligned}
& \int (J^2 + \Omega^2 + w_3^2) \phi^2 dy \Big|_0^t + \frac{1}{4} \int_0^t \int (|\nabla J|^2 + |\nabla \Omega|^2 + |\nabla w_3|^2) \phi^2 dy ds \\
& \leq (28 + 18c_2) \lambda_1 \int_0^t \int (|\nabla J|^2 + |\nabla \Omega|^2 + |\nabla w_3|^2) \phi^2 dy ds \\
& \quad + 2[\lambda_2(1 + 8c_2) + 24\lambda_1] \int_0^t \int [(J\phi)^2 + (\Omega\phi)^2] dy ds + CtS(v_0, a, l, \lambda_1, \lambda_2).
\end{aligned}$$

Here we have used the assumption that $r \leq a \leq 1$. Choosing

$$(2.36) \quad \lambda_1 = \frac{1}{4(28 + 18c_2)}.$$

Here c_2 is given in (2.13) with $q = 2$. We reduce the last inequality to

$$\begin{aligned}
& \int (J^2 + \Omega^2 + w_3^2) \phi^2 dy \Big|_0^t \\
& \leq 2[\lambda_2(1 + 8c_2) + 24\lambda_1] \int_0^t \int [(J\phi)^2 + (\Omega\phi)^2] dy ds + CtS(v_0, a, l, \lambda_1, \lambda_2).
\end{aligned}$$

By Gronwall's inequality

$$\int_{0 \leq r \leq a/2, -l/2 < y_3 < l/2} \left(\left(\frac{w_r}{r} \right)^2 + \left(\frac{w_\theta}{r} \right)^2 + w_3^2 \right) \phi^2(y, t) dy \leq C(t, v_0, a, l, \lambda_1, \lambda_2).$$

By standard theory this is more than enough to imply the regularity of v for all time. The reason is that it implies w is locally $L^{2,\infty}$ in any finite time. \square

Finally we verify the claim that v_θ is in the λ_1 critical class for any fixed $\lambda_1 > 0$, if it satisfies $|v_\theta(x, t)| \leq \frac{C}{r|\ln r|^{2+\epsilon}}$, $r < 1/2$.

Let $\psi = \psi(y, s)$ be any test function in Definition 1.1 with $a > 0$ to be specified later. Fixing s , we compute

$$\begin{aligned} \int \frac{\psi^2}{r^2|\ln r|^{2+\epsilon}} dy &= 2\pi \int_0^\infty \int_0^\infty \frac{1}{r|\ln r|^{2+\epsilon}} \psi^2 dr dy_3 \\ &= \frac{2\pi}{1+\epsilon} \int_0^\infty \int_0^\infty (|\ln r|^{-1-\epsilon})' \psi^2 dr dy_3 = -\frac{2\pi}{1+\epsilon} \int_0^\infty \int_0^\infty \frac{1}{|\ln r|^{1+(\epsilon/2)}} \frac{2\psi}{\sqrt{r}} \partial_r \psi \frac{1}{|\ln r|^{\epsilon/2}} \sqrt{r} dr dy_3 \\ &\leq \frac{2\pi}{1+\epsilon} \int_0^\infty \int_0^\infty \frac{\psi^2}{r|\ln r|^{2+\epsilon}} dr dy_3 + \frac{2\pi}{1+\epsilon} \int_0^\infty \int_0^\infty \frac{|\partial_r \psi|^2}{|\ln r|^\epsilon} r dr dy_3 \\ &\leq \frac{1}{1+\epsilon} \int \frac{\psi^2}{r^2|\ln r|^{2+\epsilon}} dy + \frac{1}{1+\epsilon} \int \frac{|\partial_r \psi|^2}{|\ln r|^\epsilon} dy. \end{aligned}$$

Therefore

$$\int \frac{\psi^2}{r^2|\ln r|^{2+\epsilon}} dy \leq \frac{1}{\epsilon|\ln a|^\epsilon} \int |\partial_r \psi|^2 dy,$$

which shows

$$\int \left(\frac{|v_\theta|}{r} + v_\theta^2 \right) \psi^2 dy \leq \frac{C + C^2}{\epsilon|\ln a|^\epsilon} \int |\partial_r \psi|^2 dy.$$

Since C , ϵ and λ_1 are fixed positive numbers, we can always choose $a > 0$ sufficiently small so that, for all $t \geq 0$,

$$\int_0^t \int \left(\frac{|v_\theta|}{r} + v_\theta^2 \right) \psi^2 dy ds \leq \lambda_1 \int_0^t \int |\partial_r \psi|^2 dy ds.$$

Therefore v_θ is in the λ_1 critical class.

Acknowledgment *The author gratefully acknowledges the supports by Siyuan Foundation through Nanjing University and by the Simons Foundation.*

He also wish to thank Prof. Lei, Zhen and Mr. Pan, Xinghong for discussions on the problem.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, RIVERSIDE, CA 92521